Gap vectors of real projective varieties

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Outline

The main actors

A polynomial $p \in \mathbb{R}[x_1, \ldots, x_m]$ is non-negative if $p(x) \geq 0$ for all $x \in \mathbb{R}^m$.

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Central question: When can a non-negative polynomial be written as a sum of squares?

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Though Hilbert did not provide an example, his proof can be used for the construction of such polynomials (Robinson, Reznick, etc.).

The Motzkin polynomial

Historically: first example of a non-negative polynomial that is not a sum of squares (Motzkin, around 1965).

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It can be shown that ...

- $M(x, y, z)$ is non-negative (arithmetic-geometric mean inequality):

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- $M(x,y,z)$ cannot be written as a sum of squares.

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Answer: **YES!** (Artin, 1927)

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Answer: **YES!** (Artin, 1927)

BUT: Degree of the multiplier may be very large.

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- $(x^2 + y^2)^2 \cdot M(x, y, z)$ can be written as a sum of squares.

$$\begin{aligned}(x^2 + y^2)^2 \cdot M(x, y, z) = &\left((x^2 - y^2)z^3\right)^2 \\ &+ \left(x^2y(x^2 + y^2 - 2z^2)\right)^2 \\ &+ \left(xy^2(x^2 + y^2 - 2z^2)\right)^2 \\ &+ \left(xyz(x^2 + y^2 - 2z^2)\right)^2\end{aligned}$$

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$\Rightarrow$ $M(x,y,z)$ can be written a sum of squares of rational functions with denominator x^2+y^2.

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- *a (multiple) cone over any of the above.*

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Theorem (Blekherman, Smith, Velasco; 2013)

Let $X \subseteq \mathbb{RP}^m$ be a non-degenerate, real irreducible projective variety.

Then $P_X = \Sigma_X$ if and only if $X(\mathbb{C})$ is a variety of minimal degree.

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Under the d^{th} Veronese embedding:

$$\begin{array}{rccc} \nu_d : & \mathbb{RP}^m & \to & \mathbb{RP}^{\binom{m+d}{d}-1} \\ & [x_0 : x_1 : \ldots : x_m] & \mapsto & [x_0^d : x_0^{d-1}x_1 : \ldots : x_m^d] \end{array}$$

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Hence: $P_{X,2d} = P_{\nu_d(X)}$ and $\Sigma_{X,2d} = \Sigma_{\nu_d(X)}$.

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If our aim is to determine the minimum

$$p^* = \min_{x \in X} p(x) = \max_{p - \lambda \in P_X} \lambda,$$

we can instead compute the approximation

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The difference between P_X and Σ_X determines the quality of the approximation.

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Outline

Setting

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In this talk: We want to determine the dimensions of generic exposed faces $P(\Gamma)$ and $\Sigma(\Gamma)$.

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Theorem (Blekherman, Iliman, J., Velasco)

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Regarding non-negative polynomials, we cannot determine the dimension of $P(\Gamma)$ for any set of points $\Gamma \subseteq X$.

We need an extra condition: independence.

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- (1) and (3) say that $\langle \Gamma \rangle$ and X intersect transversely and the intersection is Γ.

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Bottom line: A generic set of points of size $\leq c$ is independent.

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- We first show that $P(\Gamma)$ is full-dimensional in the vector space of quadratic forms vanishing to order ≥ 2 at all points of Γ.

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The gap vector measures dimensional differences between generic exposed faces of P_X and Σ_X.

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Then

$$g_\ell(X) = \epsilon(X) - \epsilon(Y).$$

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(5) If $g_{s+1}(X) - g_s(X) = c - s$ for some $s < c$, then $g_{j+1}(X) - g_j(X) = c - j$ for all $s \leq j \leq c - 1$.

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- (1) rediscovers the result by Blekherman, Smith and Velasco showing that $P_X \neq \Sigma_X$ if X is not of minimal degree.
- Not only the varieties of minimal degree (DelPezzo, Bertini) but also those with $\epsilon(X) = 1$ (Zak) are completely classified.

Veronese embeddings

- $X = \nu_4(\mathbb{RP}^2) \subseteq \mathbb{RP}^{14}$ 4^{th} Veronese embedding of $\mathbb{RP}^2$

$$[x_0 : x_1 : x_2] \mapsto [x_0^4 : x_0^3 x_1 : x_0^3 x_2 : \ldots : x_2^4]$$

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Then $\operatorname{codim}(X) = 31$ and

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d^{th} Veronese embeddings of $\mathbb{RP}^2$

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Theorem (Blekherman, Iliman, J., Velasco)

Let $X = \nu_d(\mathbb{RP}^2) \subseteq \mathbb{RP}^{\binom{d+2}{2}-1}$ be the d^{th} Veronese embedding of $\mathbb{RP}^2$. Then

$$g_j(X) = \begin{cases} 0, & \text{if } j \leq \binom{d+1}{2} \\ \left(j - \binom{d+2}{2}\right)(d-1) - \binom{j+1-\binom{d+1}{2}}{2}, & \text{otherwise.} \end{cases}$$

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The growth in each step is extremal.

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Question:

What about gap vectors of general Veronese embeddings of $\mathbb{RP}^m$?

d^{th} Veronese embeddings of $\mathbb{RP}^m$

Conjecture (Blekherman, Iliman, J., Velasco)

Let $X = \nu_d(\mathbb{RP}^m)$. *Let*

$$j^* = \left\lceil \binom{n+d}{d} - (n+1) + \frac{1}{2} - \sqrt{(n+\frac{1}{2})^2 + 2\binom{n+2d}{2d} - 2(n+1)\binom{n+d}{d}} \right\rceil .$$

Then

(1) $g_j(X) = 0$ *for* $1 \leq j < j^*$,

(2) $g_j(X) = \binom{m+2d}{2d} - j(m+1) - \binom{\binom{m+d}{d} - j + 1}{2}$, *for* $j^* \leq j \leq \operatorname{codim}(X)$.

Thank you for your attention!